1990MNRAS.245..733A

Mon. Not. R. astr. Soc. (1990) **245,** 733–746

Centrifugal force: a few surprises

Marek A. Abramowicz★
International Centre for Theoretical Physics Trieste, Italy and Max-Planck-Institut für Physik und Astrophysik, Institut für Astrophysik, Garching bei München, FRG

Accepted 1990 March 21. Received 1990 February 21; in original form 1989 July 3

SUMMARY

The need for a rather fundamental revision in understanding of the nature of the centrifugal force is discussed. It is shown that in general relativity (and contrary to the situation in Newtonian theory) rotation of a reference frame is a necessary but not sufficient condition for the centrifugal force to appear. A sufficient condition for its appearance, in the instantaneously corotating reference frame of a particle, is that the particle motion in space (observed in the global rest frame) differs from a photon trajectory. The direction of the force is the same as that of the gradient of the effective potential for photon motion. In some cases, the centrifugal force will *attract* towards the axis of rotation.

1 INTRODUCTION

Some very strange dynamical effects of rotation in strong graviational fields have been discussed for a long time within the framework of general relativity. Several of them are clearly connected with *frame dragging*, i.e. the rotation of locally defined 'inertial' reference frames with respect to the globally defined rest frame. The local frames can be determined by reference to a set of gyroscopes while the global one is determined by reference to observations of distant galaxies. Frame dragging which occurs in the gravitational field of spinning bodies can influence the dynamics of rotation in ways which are sometimes quite unexpected. Because of this, there is a tendency to try to explain *all* strange rotational effects as being due to frame dragging. This tendency is sometimes counter-productive: some proposed explanations for the dynamical effects of rotation have offered little physical insight but, instead, have just introduced synonymous descriptions of the same thing, changing the phrase 'frame dragging' into, for example, 'coupling between the angular momentum of the matter and of the black hole' or 'spin–spin interaction', or other expressions of a similar kind. However, it is clear that many of the most bizarre rotational effects have nothing to do with frame dragging. For example, Abramowicz & Lasota (1974, 1986, hereafter AL) noticed that test particles (all with identical mass m) orbiting on a circle $r=3GM/c^2$ around a Schwarzschild black hole (with mass M) all have to use the same rocket thrust $\mathscr{T}=mc^4/6GM$ in order to stay on the orbit, irrespective of their orbital speed v. This is very different from Newtonian theory which suggests the need for a *velocity-dependent* thrust in order to maintain the circular motion, $\mathscr{T}=(v_K^2-v^2)/r$. (Here $v_K=(GM/r)^{1/2}$ is the Keplerian orbital velocity.) The difference from Newtonian theory cannot be explained in terms of frame dragging, because no frame dragging occurs in the Schwarzschild space-time. Another example of a strange rotational effect which does not depend on frame dragging can be found in a recent paper by Anderson & Lemos (1988): close to a Schwarzschild black hole, viscous stresses in thin accretion discs transport angular momentum *inwards*. This is contrary to a well-known result from the classical theory of thin accretion discs that viscous stresses always transport angular momentum outwards (Lynden-Bell & Pringle 1974). Frame dragging is also not relevant for another effect, found by Abramowicz & Prasanna (1990), that close enough to a Schwarzschild black hole ($r<3GM/c^2$), the Rayleigh stability criterion is reversed: stable equilibria correspond to angular momentum *decreasing* outwards, whereas the conventional Rayleigh criterion demands that the angular momentum must *increase* outwards for stability.

In all of the examples mentioned above, self-gravity of the rotating matter was ignored. Turning now to examples of strange rotational effects in the dynamics of self-gravitating, rotating bodies, we recall the important result of Chandrasekhar & Miller (1974) and Miller (1977) who discovered that the ellipticity of quasi-stationarily contracting relativistic Maclaurin spheroids with fixed mass M and total angular momentum J *decreases* with decreasing mean radius R when $R<5GM/c^2$. This is in contrast to Newtonian theory which predicts that, when a rotating body shrinks conserving angular momentum, it always becomes progressively more flattened. Abramowicz & Wagoner (1976) found a similar effect: rotation *increases* internal pressure of a sufficiently compact body (having $R<2.5\ GM/c^2$).

★Present address: Scuola Internazionale Superiore di Studi Avanzati, Strada Costiera 11 34014 Trieste, Italy.

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Until recently, there was no satisfactory explanation for the cause of all these strange effects (and other similar ones). The proposed explanations were complicated, particular and different in each case. Many of them missed the point claiming, incorrectly, that the reason for the strange dynamical properties of rotation mentioned above, was either frame dragging itself or another effect connected with spin of the source of the gravitational field.

In this paper I describe a full, consistent, unique and simple explanation for the effects of rotation in a strong gravitational field. My explanation is based on a discussion of the nature of the centrifugal force. Although I use the most conservative and conventional approach, the conclusion of the discussion is quite unexpected: I claim, that the rotational effects are indeed bizarre – they can be properly understood only after a fundamental revision of the very concept of centrifugal force. The most important element of this revision is that the centrifugal force should not be connected to rotation of a reference frame, but to motion along a *curved path* in space. 'Curved path' means a path different from the space-like projection of a null geodesic, i.e. different from that of a light ray (a photon path in space). Another unexpected conclusion from this discussion is that, under some circumstances, *the centrifugal force can attract matter towards the axis of rotation.*† These statements are justified not only formally but also in terms of several *gedanken experiments* based on the fact that the centrifugal force can actually be *measured* independently of whether one is using Newtonian or relativistic dynamics. Thus they have precise experimental meanings and can be experimentally verified.

My explanation uses results from the recent papers by Abramowicz, Carter & Lasota (1988, hereafter ACL), Abramowicz & Prasanna (1990, hereafter AP) and Abramowicz & Miller (1990, hereafter AM). The formal basis for all of these papers (and for the present one) was given by ACL who demonstrated that there is a unique, covariant, *geometrical* way of defining centrifugal force in general relativity. They carried out their discussion using the *optical reference geometry*‡ which is a rather formal relativistic construction whose logical structure and geometrical appeal are probably unlikely to impress non-specialists. I will not use the optical geometry in this paper, but will make some comments about its relevance for understanding the dynamical effects of rotation.

The two papers (AP and AM), which accompany this one in the present issue of Monthly Notices, contain discussion of several particular astrophysical examples connected with the reversal of centrifugal force. They were both completed before I wrote this paper. I would like to emphasize that the possibility that the centrifugal force can reverse direction was first anticipated (on physical grounds) by Abramowicz & Lasota (1974, 1986). In those papers, we claimed that the centrifugal force should be zero on the circular photon orbit and this was later supported by a *formal* discussion in the ACL paper. The explicit demonstration that the centrifugal force can reverse its direction and attract matter towards the rotation axis, was given in the AP paper where the formal approach of ACL was used.

† In this case it might be better to refer to the force as 'centripetal', but I prefer to speak about negative 'centrifugal' force. Newton, who introduced the concept of 'centripetal force' chose this name because it was similar (but opposite) to Huygens' 'vis centrifuga' (Cohen 1985).

‡This has nothing in common with several 'optical geometries' or 'optical coordinates' discussed previously in a different context – see e.g. Synge (1960) or, more recently, Trautman (1984) and Robinson & Trautman (1986). Using the name 'optical geometry' for different and unrelated concepts is already an established tradition in general relativity: see, for example, the footnote on page 86 of Synge's (1960) textbook which explains how the 'optical coordinates' defined in his book differ from those used earlier by Temple (1938).

2 TWO CONCEPTS OF CENTRIFUGAL FORCE

According to Isaac Newton's point of view, clearly explained in *Principia* by his famous experiment with a rotating pail of water (see Weinberg 1972), the centrifugal force $\mathbf{C}$ is a force which acts on matter in a rotating frame of reference and whose magnitude and direction are *not* connected with the motion of the matter, but are fully determined by the rotation of the reference frame. If the angular velocity of the rotating frame, measured with respect to *absolute space*, is $\mathbf{\Omega}$, and the distance of a particle (with mass m) from the axis of rotation of the frame is $\mathbf{r}$, then the centrifugal force acting on this particle *in the rotating frame* is given by

$$\mathbf{C} = m\mathbf{\Omega} \times (\mathbf{r} \times \mathbf{\Omega}). \tag{2.1a}$$

I will call (2.1) *Newton's definition* of the centrifugal force. According to this, the centrifugal force *always* acts in the direction of the vector $\mathbf{r}$ which can be geometrically defined in the following way. In a time $t = 2\pi/\Omega$ (with $\Omega = |\mathbf{\Omega}|$) all fixed points in the rotating frame make full circles in the global rest frame and, in particular, the particle in question describes a circle of circumferential length s_0. All of the circles with $s = s_0$ together form a cylindrical surface, whose axis coincides with the axis of rotation. Let $r = s/2\pi$ and let $\mathbf{q}$ be the unit vector normal to the cylindrical surfaces (pointing outwards):

$$\mathbf{q} = \frac{\mathbf{\nabla} r}{|(\mathbf{\nabla} r)\cdot(\mathbf{\nabla} r)|^{1/2}}. \tag{2.1b}$$

The vector $\mathbf{r}$ can be written as

$$\mathbf{r} = r\mathbf{q}. \tag{2.1c}$$

The speed of the particle, measured in the rest frame, is $v = \Omega r$ and from this it follows that, according to Newton's definition, the centrifugal force is given by

$$\mathbf{C} = \frac{v^2}{r}\mathbf{q}. \tag{2.1d}$$

In the modern formulation of Newtonian dynamics the expression 'rotation with respect to absolute space' is not used. Instead, rotation is defined using an inertial frame or a *global rest frame*, as in this paper. The question of what determines the global rest frame (absolute space itself?, all of the matter in the Universe?) has been discussed by Berkeley, Newton, Leibnitz, Mach, Einstein and others and is still disputed today [see Sciama (1959) for an historical account and an explanation of modern understanding]. Although this question is relevant in the present context, I avoid it here by assuming that, for whatever reason, a global rest frame *does* exist. Technically, in general relativity, the global rest frame is

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associated with time symmetry of the space-time and is invariantly defined by that Killing vector which is time-like near infinity. Note that the existence of the global rest frame can, at least in principle, be determined experimentally. Consider a family of observers distributed in space (with or without the presence of a gravitational field). Each observer is equipped with a test rocket engine, a clock, a radar emitter and receiver, and three theodolites with gyroscopically guided axes. Each observer continually measures distances to all of the other observers by means of the radar time-delay technique. He also continually measures, by means of the theodolites, the locations of all of the other observers in his celestial sphere and he continually monitors the strength and direction of the thrust of his rocket engine. If, by changing the direction and strength of thrust of their rockets, the observers can achieve a situation in which all of their measurements (distances, directions, thrusts, thrust directions) no longer depend on time as measured by their clocks, then these observers will have established a global rest frame (compared this with the discussion given by Ehlers 1973).

According to the above definition, a global rest frame can only coexist with a gravitational field if that field is static since, otherwise, the thrusts needed to keep observers 'in the same places' will obviously depend on time. In a general static gravitational field, the global rest frame is unique.§ The global rest frame does *not* coincide with the *locally* defined inertial frames (freely falling Einstein elevators) in which the gravitational force is zero, but is identical with a particular (global) inertial frame in which gravity is non-zero. Since Newtonian dynamics can be fully discussed using inertial frames, one might argue that there is no need for the concept of centrifugal force. Indeed, the classical monograph by Landau & Lifshitz (1976), devoted to Newtonian dynamics, quotes the term 'centrifugal force' only once in the whole 166 pages of the book. One may argue even more strongly (using 'obvious' and seemingly convincing arguments) against introducing it into general relativity.

However, the concept is very useful because it appeals strongly to physical intuition. Our everyday experience with centrifugal force (on which our intuition seems to be based) comes from feeling its effect in trains, cars, and planes when they are making turns. In each of these cases we *corotate* with the frame connected with the turning vehicle and thus we feel the centrifugal force as described by formulae (2.1). Because of this experience and intuition it is both natural and easy to imagine the effect of the centrifugal force which acts on people who are travelling on a high-speed roller-coaster train (say) even if we observe them from the safety of the global rest frame in which the centrifugal force is absent. Our intuition quite correctly infers the most relevant dynamical feature of the motion: the stronger the curvature of the tracks and the faster the speed of the roller-coaster train, the stronger the outward push of the centrifugal force. In this intuitive picture, we translate the real features of the situation (curvature of the tracks, speed of the train) which we see and which we could measure in the global rest frame, into those which refer to an abstract frame *instantaneously corotating* with the roller-coaster passengers. It is not difficult to see that *we use exactly the same intuition when talking about physics*. A good example of a practical use of this intiution is the sentence: 'In circular planetary motion the gravitational force acting on a planet, $\mathscr{G} = -GMm/r^2$, balances the centrifugal force $\mathscr{C} = m\Omega^2 r$ and from the balance condition $\mathscr{G} + \mathscr{C} = 0$ one can compute the Keplerian orbital angular speed $\Omega_K = (GM/r^3)^{1/2}$'. The physical sense of this sentence seems to be very precise and clear and its meaning seems to be correct. However, the sentence refers to quantities defined in the global rest frame where the centrifugal force does not appear! Obviously, our intuition performs a transformation between the global rest frame and several rotating ones (different for each planet). This allows us to understand the dynamic situation in the instantaneously corotating reference frames in terms of quantities defined in the global rest frame. It seems that we always intuitively analyse dynamical effects of rotation in the corotating reference frames.¶

To see how the 'roller-coaster' intuition agrees with Newton's definition of centrifugal force, let us consider, *in a global rest frame*, a fixed curve (the 'roller-coaster tracks') $x = \mathbf{x}(S)$ and a particle (the 'roller-coaster train') moving with a prescribed speed $v = v(S)$ along the curve. (Distance along the curve is denoted by S.) The idea is to compute the centrifugal force $\mathbf{C}$ acting on the particle in its own *instantaneously corotating reference frame* at each point of the curve and to express the force in terms of quantities defined in the global rest frame. First, let us define the instantaneously corotating reference frame using only the functions $\mathbf{x}(S)$ and $v(S)$. There is only one correct way to do this and it is based on the Frenet formulae (e.g. Synge & Schild 1959). The Frenet formulae define at each point of the curve $\mathbf{x}(S)$ a set of three mutually orthogonal unit vectors – the tangent vector $\boldsymbol{\tau}$, the first (outward) normal $\boldsymbol{\lambda}$ and the second normal $\boldsymbol{\Lambda}$:

$$\boldsymbol{\tau} = \frac{dx}{dS}, \qquad v = \frac{dS}{dt}, \qquad \mathbf{v} = v\boldsymbol{\tau}, \tag{2.2a}$$

$$\boldsymbol{\lambda} = -\mathscr{R}(\boldsymbol{\tau}\cdot\boldsymbol{\nabla})\,\boldsymbol{\tau}, \tag{2.2b}$$

$$\boldsymbol{\Lambda} = \mathscr{S}\left(\frac{d\boldsymbol{\lambda}}{dS} + \frac{1}{\mathscr{R}}\boldsymbol{\tau}\right) = \boldsymbol{\lambda}\times\boldsymbol{\tau}. \tag{2.2c}$$

Here $\boldsymbol{\nabla}$ is the covariant space derivative, $\mathscr{R}$ is called the geodesic-curvature radius (or first curvature) and $\mathscr{S}$ is called the torsion (or second curvature). The vectors $\boldsymbol{\tau}$, $\boldsymbol{\lambda}$, $\boldsymbol{\Lambda}$ are called the Frenet triad and they obey the relations

$$\boldsymbol{\tau} = \boldsymbol{\Lambda}\times\boldsymbol{\lambda},\ \boldsymbol{\lambda} = \boldsymbol{\tau}\times\boldsymbol{\Lambda},\ \boldsymbol{\Lambda} = \boldsymbol{\lambda}\times\boldsymbol{\tau}. \tag{2.2d}$$

Note that different sign conventions are often used for the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$; in particular the vector $\boldsymbol{\lambda}$ is often taken with the opposite sign. We have chosen the present particular convention because it is the most appropriate one for the discussion of centrifugal force.

§ This is untrue only in the pathological case of a uniform gravitational field (which also includes the case of zero gravity).

¶ Nothing proves better than our intuition really does work in this way than a puzzle told to me by Dennis Sciama. Consider the simplest possible situation – a particle at rest in the global rest frame and an observer who uniformly rotates at some distance. The observer will see a centrifugal force $\mathbf{F}$ acting on the particle and directed, obviously, *outwards* with respect to the particle's circular trajectory in his rotating frame. He will measure, however, that the centripetal acceleration $\mathbf{a}$ is directed *inwards*, as it always is in circular motion. Therefore $\mathbf{F}$ and $\mathbf{a}$ point in opposite directions, and Newton's law $\mathbf{F} = m\mathbf{a}$ is not obeyed. What is wrong?

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When the geodesic curvature radius is infinite, $\mathscr{R}=\infty$, the curve $\mathbf{x}(S)$ is a straight line and it is not possible to define the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$. However, later in my discussion I will need to define directions orthogonal and tangent to a trajectory and so I will introduce an *arbitrary* unit vector $\boldsymbol{\lambda}^*$, orthogonal to the curve, which is not defined in any special way when $\mathscr{R}=\infty$, but is a linear combination of $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ when $\mathscr{R}\neq\infty$:

$$\boldsymbol{\lambda}^*=a\boldsymbol{\lambda}+b\boldsymbol{\Lambda}, \tag{2.2e}$$

$$a^2+b^2=1. \tag{2.2f}$$

The instantaneously corotating reference frame for a particle moving along the curve $\mathbf{x}(S)$ with speed $v(S)$, is defined in terms of the angular velocity of the frame $\boldsymbol{\Omega}$ and the displacement $\mathbf{r}$ of the particle from the instantaneous rotation axis. These two vectors are given by the following formulae:

$$\boldsymbol{\Omega}=\left(\frac{v}{\mathscr{R}}\right)\boldsymbol{\Lambda}, \tag{2.3a}$$

$$\mathbf{r}=\mathscr{R}\boldsymbol{\lambda}. \tag{2.3b}$$

From Newton's definition of the centrifugal force (2.1), the definition of the instantaneously corotating reference frame (2.3) and the Frenet formulae (2.2), one can derive the expression giving the centrifugal force in the instantaneously corotating reference frame in terms of quantities defined in the global rest frame (i.e. according to the 'roller-coaster' intuition):

$$\mathbf{C}\equiv m\boldsymbol{\Omega}\times(\mathbf{r}\times\boldsymbol{\Omega})=m\frac{v^2}{\mathscr{R}}\boldsymbol{\lambda}. \tag{2.4}$$

I will call (2.4) *Huygens' definition* of centrifugal force.** In the Newtonian theory, *based on Euclidean geometry*, the Newton and Huygens definitions are equivalent, because $\mathscr{R}=r$ and $\boldsymbol{\lambda}=\mathbf{q}$. However, they stress physically *different* aspects of the centrifugal force: the first one speaks of rotation of the reference frame with respect to the global rest frame, while the second one refers to motion along a curved path in space ($\mathscr{R}\neq\infty$). When the space is not Euclidean, these definitions are *not* equivalent. To see this, let us consider a uniformly rotating frame A and a test particle which is at rest with respect to it. According to Newton's definition, there will always be a non-zero centrifugal force acting on this particle in the rotating frame A. Consider now the particle's circular path in the global rest frame. In a particular non-Euclidean geometry it may happen that the geodesic-curvature radius of this path is infinite and so, according to Huygen's definition, there would be no centrifugal force acting on the particle in the same rotating frame A! Therefore, it is obvious that a general relativistic definition of centrifugal force cannot be consistent with both Newton's and Huygens' definitions. By discussing a *gedanken experiment*, in which the centrifugal force can be measured, I will show later that, in a non-Euclidean space, Huygen's definition is physically correct and not the Newtonian one.

** A formula equivalent to (2.4) was derived by Christiaan Huygens in his treatise *De vi centrifuga* written in 1659 but only published in 1703 after his death. In 1669 he established his claim to authorship of the formula by sending it, in anagrammatic form, to Oldenburg who was then the secretary of the Royal Society. The solutions of the anagrams were given in Huygens' book *Horologium Oscillatorium* which was published in Paris in 1673 (Thorndike 1958). The use of the term 'Huygens' definition' for expression (2.4) also seems appropriate because of the particular view which he at first took of circular motion. Taten (1958) says: "At first he contrasted uniform and rectilinear (*straight*) motion with circular motion: *Whereas straight motion is relative, circular motion has a characteristic χριτήριον of his own.* This 'criterion' was evidently the tension of strings under the centrifugal force of revolving bodies attached to them. However, when he read Newton's *Principia*, he reacted so violently that he adopted a more strictly Cartesian view: *One can in no wise conceive of the true and natural motion of a body as differing from its [state of] rest… In circular motion as in free and straight motion, there is nothing except relativity.*"

I will conclude this section with a short comment on the four reference frames which are most natural for discussing the dynamics of rotation about a fixed axis: (i) the global rest frame, (ii) the non-rotating, freely falling frame, (iii) the corotating frame, and (iv) the corotating, freely falling frame. The equation of motion in each of these frames contains *the same* three terms: 'applied' force, gravity and rotation, but the physical interpretation of these terms is difficult in the different frames. Let $\boldsymbol{\Phi}$ denote the gravitational potential, $\mathbf{g}=-\boldsymbol{\nabla}\boldsymbol{\Phi}$, the acceleration of the particle in the freely falling frame, and $\mathbf{G}=m\boldsymbol{\nabla}\boldsymbol{\Phi}$, the gravitational force. Centrifugal force is denoted, as usual, by $\mathbf{C}$ and centripetal acceleration by $\mathbf{c}=-(1/m)\mathbf{C}$. The applied force, denoted by $\mathbf{T}$, is due to mechanical stresses from ropes, springs, rocket engines (thrust) etc., or other stresses (electromagnetic, for example), and is necessary in order to keep the particle moving along a non-straight and non-Keplerian trajectory, $\mathbf{x}(S)$. It acts in a direction orthogonal to the trajectory and is often referred to as 'constraining'. For my arguments, it is not important which kind of constraining force is present but, in this paper, I will always refer to just one possibility – a thrust from a rocket engine. Nothing in my arguments changes however, if the thrust is replaced by tension of a spring, solid body support, etc.

I will write the equation of motion, $\mathbf{F}=m\mathbf{a}$, in each of the four different frames, keeping all of the forces on the left-hand side and all of the accelerations on the right-hand side. The equation of motion in the global rest frame has the form

$$\mathbf{T}+\mathbf{G}=m\mathbf{c}, \tag{2.5}$$

while in the non-rotating, freely falling frame, it is

$$\mathbf{T}=m(\mathbf{g}+\mathbf{c}). \tag{2.6}$$

In the corotating frame, the total acceleration vanishes

$$\mathbf{T}+\mathbf{G}+\mathbf{C}=0. \tag{2.7}$$

and in the freely falling, corotating frame:

$$\mathbf{T}+\mathbf{C}=m\mathbf{g}. \tag{2.8}$$

There is a subtle but very important point which should be explained here, connected with different meanings of the terms $\mathbf{F}$ and $\mathbf{a}$ in the equation of motion (which in both Newtonian theory and general relativity has the same form $\mathbf{F}=m\mathbf{a}$). In Newtonian theory, $\mathbf{a}=0$ for a particle which always stays at a fixed point in a gravitational field and $\mathbf{a}\neq0$ for a freely falling particle. In general relativity exactly the reverse is true. (This is because, in Newtonian theory, 'gravity' is treated as a force while in general relativity it is taken as part of the acceleration.) The Newtonian equation of motion in the freely falling frames (2.6) and (2.8) corresponds most closely, therefore, to the relativistic equation of motion in the rest frames. In Section 4, I will use the relati-

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vistic global rest frame, in which the equation of motion closely resembles equation (2.7) above.

3 GEDANKEN EXPERIMENTS TO MEASURE THE CENTRIFUGAL FORCE

In this section, I will discuss how to measure the centrifugal force *experimentally* when a global rest frame exists. The experiments will be performed in exactly the same way in relation to Newtonian and relativistic dynamics and, for this reason, the measured quantities and experimental procedures will not depend on concepts (e.g. gravity) which are well defined only in Newtonian theory and have no precise meaning in general relativity. However, in order not to make the discussion too technical, I will describe the experiments here using only Newtonian terminology. In the subsequent section all of the necessary technical details will be fully explained according to general relativity.

The centrifugal force is not the only apparent force present in a general rotating reference frame. Two other two forces are also present:

$$\mathbf{C}^{**} = 2m\mathbf{v}_0 \times \mathbf{\Omega}, \tag{3.1a}$$

$$\mathbf{C}^{***} = \dot{\mathbf{\Omega}} \times \mathbf{r}, \tag{3.1b}$$

the first of which is the Coriolis force. In the formulae (3.1), $\mathbf{v}_0$ is the velocity of the particle in the rotating frame and $\dot{\mathbf{\Omega}}$ is the rate of change of the rotation speed of the frame. From equations (2.2) and (2.3) it follows that, in the instantaneously corotating reference frame,

$$\mathbf{v}_0 = \mathbf{v} - \mathbf{\Omega} \times \mathbf{r} = \mathbf{v} - v(\mathbf{\Lambda} \times \boldsymbol{\lambda}) = 0, \tag{3.2a}$$

$$\mathbf{C}^{\dagger} = 0. \tag{3.2b}$$

and so *the Coriolis force is identically zero in the instantaneously corotating reference frame.*†† It is also easy to see that the force $\mathbf{C}^{***}$ acts in a direction parallel to the vector $\boldsymbol{\tau}$ and therefore, for *any* unit vector orthogonal to the trajectory,

$$\mathbf{C}^{***} \cdot \boldsymbol{\lambda}^* = 0. \tag{3.3}$$

Let us now discuss the equation of motion (the second law of Newtonian dynamics):

$$\mathbf{F} = m\mathbf{a}. \tag{3.4}$$

The acceleration in the instantaneously corotating reference frame vanishes in any direction $\boldsymbol{\lambda}^*$ normal to the trajectory ($\mathbf{a} \cdot \boldsymbol{\lambda}^* = 0$) and so does the total force acting on the particle ($\mathbf{F} \cdot \boldsymbol{\lambda}^* = 0$). The total force consists of the applied force (thrust) $\mathbf{T}$, the gravitational force $\mathbf{G}$ and the apparent forces introduced by use of the reference frame.

The component $\mathcal{T}^*$ of the thrust in the direction of an arbitrary (but fixed) unit vector $\boldsymbol{\lambda}^*$ orthogonal to the trajectory,

$$\mathcal{T}^* \equiv \mathbf{T} \cdot \boldsymbol{\lambda}^*, \tag{3.5}$$

is a clearly defined quantity in both Newtonian theory and general relativity. It can be experimentally measured. When there is a global rest frame, one can experimentally measure the speed v of different particles moving along the trajectory and, hence, the quantity

$$\mathcal{X}^* = -v^2 \left(\frac{\partial \mathcal{T}^*}{\partial v^2} \right). \tag{3.6}$$

Following from equation (2.7) above (for forces in the instantaneously corotating reference frame in Newtonian theory):

$$\mathcal{T}^* + \mathcal{G}^* + \mathcal{C}^* = 0. \tag{3.7}$$

Since the $\boldsymbol{\lambda}^*$ component of the centrifugal force, $\mathcal{C}^*$, satisfies

$$\mathcal{C}^* = v^2 \left(\frac{\partial \mathcal{C}^*}{\partial v^2} \right), \tag{3.8}$$

and the gravitational force does not depend on the speed, it follows that the experimentally measured quantity $\mathcal{X}^*$ should be identified with $\mathcal{C}^*$. (By changing the vector $\boldsymbol{\lambda}^*$ one can find the complete vector $\mathbf{C}$, and hence the whole centrifugal force can be experimentally measured.) This operational definition of centrifugal force turns out also to be appropriate within the framework of general relativity.

Suppose that we have experimentally checked, using the method embodied in formula (3.6), that $\mathcal{X}^* \equiv 0$ along a particular curve $\mathbf{x} = \mathbf{x}_0(S)$, independently of the speed and of the choice of $\boldsymbol{\lambda}^*$. In Newtonian theory, as we have seen, $\mathcal{X}^* = \mathcal{C}^*$ is the centrifugal force measured in the instantaneously corotating frame and therefore, according to Huygen's definition (2.4), the curve $\mathbf{x}_0(S)$ must have an infinite geodesic-curvature radius (i.e. it must be a straight line). This means that $\mathbf{x}_0(S)$ has two other dynamical properties which can be *experimentally* checked:

(i) the curve $\mathbf{x}_0(S)$ coincides with a light trajectory in space;
(ii) a gyroscope moved along the curve $\mathbf{x}_0(S)$ will not precess.

In the next section, I will show that exactly the same is true for static space-times in general relativity: a curve $\mathbf{x}_0(S)$, defined by the experimental condition $\mathcal{X}^* = 0$, indeed coincides with a light trajectory and has the property that a gyroscope moved along it does not precess. Thus, $\mathcal{X}^*$ has the same dynamical properties in both Newtonian theory and general relativity and consequently it should be considered as the experimentally defined centrifugal force in both of these theories. Such a definition captures the most important physical property of the centrifugal force: the force appears (in the instantaneously corotating frame) when the motion of a particle deviates from a 'straight line' as established dynamically by light tracing and gyroscopic guiding.††

††However, the frame of Sciama's puzzle (*cf.* footnote on p. 735) is *not* a corotating frame and the Coriolis force is non-zero in it. The total force in the puzzle is the combination of the centrifugal force (pointing outwards) and the Coriolis force (pointing inwards) which is twice as large. The second law of Newtonian dynamics *is* then obeyed. Our intuition (incorrectly) ignored the Coriolis force in the puzzle because, as I have explained, we always tend to think in terms of the instantaneously corotating frame (where the Coriolis force is absent).

††In Newtonian theory the dynamically straight line is also straight in the geometrical sense: namely it is a geodesic in space. Note also that, in stationary but non-static space-times, the dragging of inertial frames causes the two standards of dynamically straight motion to be different: light does not propagate along the lines connected with transport of non-precessing gyroscopes. ACL have argued that (formally) this can be described as a manifestation of a Coriolis force. Chakrabarti & Prasanna (1989, poster at the GR12 Conference) pursued further this formal approach of ACL and discussed in great detail the properties of such a formally defined Coriolis force in the Kerr geometry.

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I will next describe an experiment to determine whether the centrifugal force repels or attracts towards the rotation axis. Suppose that we have a circular ring (made of steel or another very strong material), with radius $r=r_0$, and that attached to it are several strings, much shorter than the radius, which have a finite strength and can resist only an internal tension less than some critical value $\mathscr{T}_0$. We set up a cylindrical coordinate system, at the origin of which ($r=0$, $z=0$) is placed a gravitating point mass M, and the ring is moved along the axis of symmetry in such a way that its centre always coincides with the axis and its plane is always orthogonal to the axis. We next rotate the ring with some fixed angular velocity Ω. Very far from the origin, at $z \gg r_0$, the gravitational force is negligible and therefore the strings will point exactly in the outward direction away from the axis of rotation, i.e. in the direction of the vector **q**. We now move the ring towards the origin and see that the strings point to a direction which is neither that of **q**, nor of $-$**q**. However, when the ring arrives finally at $z=0$, the strings will point either in the direction exactly outwards away from the axis of rotation, **q**, or exactly inwards towards the axis, $-$**q**. (We do not consider the 'measure zero' possibility that the ring will arrive at $z=0$ with exactly Keplerian velocity.) Suppose that the strings point towards the axis (the opposite possibility is easy to discuss in a symmetric way) and that the critical value of the tension $\mathscr{T}_0$ is almost reached so that the strings could not resist even an infinitesimal further increase in the tension. We now ask the following questions: should we speed up or slow down the rotation of the ring in order to break the strings?

Note, that the tension is the sum of the gravitational and centrifugal forces, $\mathscr{T}=|\mathscr{G}+\mathscr{C}|$. Although in general relativity there is no obvious way to divide the sum $\mathscr{C}+\mathscr{G}$ into gravity and centrifugal force, we can assume that the gravitational force always *attracts* to the centre – in Newtonian theory and general relativity alike. When the gravitational field is spherically symmetric, it is easy to see that this must be the case. Suppose, that the gravitational force were to 'change sign' due to some 'repulsive phenomenon'. If this happened for one particular ring location, then gravity (by symmetry) ought to be repulsive on the whole sphere with radius r_0 and this would imply (by the Gauss theorem) that the mass M should be negative, which is impossible. Therefore, the gravitational force is always attractive and the question of how to break the strings depends on the direction in which the centrifugal force acts. If it repels away from the axis, we should *slow down* the rotation of the ring in order to break the strings whereas if the centrifugal force attracts towards the axis, we should *speed up* the rotation in order to produce this result. Thus, the *Ring & String* experiment described above provides a 'yes-or-no' check to find out in which direction the centrifugal force acts:

$$\begin{pmatrix}\text{ring should be}\\ \text{slowed down}\\ \text{to break the strings}\end{pmatrix} \Rightarrow \begin{pmatrix}\text{centrifugal force}\\ \text{repels away}\\ \text{from the axis}\end{pmatrix},$$

$$\begin{pmatrix}\text{ring should be}\\ \text{speeded up}\\ \text{to break the strings}\end{pmatrix} \Rightarrow \begin{pmatrix}\text{centrifugal force}\\ \text{attracts}\\ \text{towards the axis}\end{pmatrix}.$$

4 CIRCULAR MOTION IN STATIC, AXIALLY SYMMETRIC SPACE-TIMES

This section consists of three parts. In the first part I give all of the formal definitions and calculations which are necessary for deriving results concerning the centrifugal force in a strong gravitational field. The first part is addressed to those readers who would like to check the details of the relativistic formalism. Those who are not so interested in the formalism should skip the first part and start reading this section from the second part where I derive most (but not all) of the results again using physical arguments and no relativistic calculations.

4.1 Formal approach

I will adopt the space-like signature convention $(-\,+\,+\,+)$ and so time-like vectors will have negative lengths and space-like vectors will have positive lengths. I will also adopt units in which the speed of light and the gravitational constant are equal to one. The Greek letter indices run through 0, 1, 2, 3; the covariant derivative in the space-time is denoted by ∇_α and the scalar product of two vectors x^α and y_α is denoted by $x^\alpha y_\alpha \equiv (xy)$.

I will make the assumption that the space-time is static and axially symmetric, a statement which has a precise geometrical meaning. First of all, it means that there exist two Killing vector fields in the space-time: the time-like Killing vector η_α, which has open trajectories, and the space-like Killing vector ξ_α, which has closed ones. The two vectors η^α and ξ^α are orthogonal

$$\eta^\alpha \xi_\alpha \equiv (\eta\xi) = 0, \tag{4.1a}$$

obey the Killing equation

$$\nabla_{(\alpha}\eta_{\beta)} = 0 = \nabla_{(\alpha}\xi_{\beta)}, \tag{4.1b}$$

and commute

$$\xi^\alpha \nabla_\alpha \eta^\beta = \eta^\alpha \nabla_\alpha \xi^\beta. \tag{4.1c}$$

For any axially symmetric and stationary scalar X one has

$$\eta^\alpha \nabla_\alpha X = 0 = \xi^\alpha \nabla_\alpha X. \tag{4.1d}$$

Finally, for *any* commuting Killing vectors η^α and ξ^α (note that a Killing vector always commutes with itself) one has

$$\eta^\alpha \nabla_\alpha \xi_\beta = -\tfrac{1}{2}\nabla_\beta(\eta\xi). \tag{4.1e}$$

The equations (4.1a–e) will be used frequently in the calculations described in this section.

Let us start by discussing the invariant definition of the outward direction with respect to the symmetry axis. The proper circumferential radii of the circles which are trajectories of the group of motions generated by the axial symmetry (i.e. by the Killing vector ξ^α) are given by

$$r = \sqrt{(\xi\xi)}. \tag{4.2}$$

The symmetry axis itself is characterized by $r=0$ and the 'straight' coaxial cylinders§§ are defined, as in Newtonian

§§ We defined the Newtonian equivalent of these straight coaxial cylinders at the beginning of Section 2 and showed that the centrifugal force acts in the direction given by the unit-(outward-pointing) normal vector to these cylinders **q**.

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theory by $r=$ constant. The unit-vector normal to these cylinders,

$$q_\alpha=\frac{\nabla_\alpha r}{[(\nabla_\beta r)(\nabla^\beta r)]^{1/2}}, \tag{4.3}$$

determines the outward direction with respect to the symmetry axis. Note, that the above formula is identical to the Newtonian formula (2.1b).

Due to space-time symmetries connected with the Killing vectors η^α and ξ^α, there are two constants for a general geodesic motion with four velocity v^α: the energy $\mathscr{E}=-\eta^\alpha v_\alpha$ and the angular momentum $\mathscr{L}=\xi^\alpha v_\alpha$. It is convenient to define the specific angular momentum

$$l=\frac{\mathscr{L}}{\mathscr{E}}, \tag{4.4}$$

which is also a constant for a general geodesic motion

$$v^\alpha\nabla_\alpha l=0. \tag{4.5}$$

Now we introduce the concept of circular motion of test particles with test rocket engines, similar to that used for the discussion within Newtonian theory presented in the previous section. The most general case of circular motion of a particle (or fluid) with four velocity v^α in a static, axially symmetric space-time is characterized by (see Boyer 1965, 1966; Carter 1969, 1973; Bardeen 1973; Abramowicz 1971, 1974)

$$v^\alpha=A(\eta^\alpha+\Omega\xi^\alpha). \tag{4.6}$$

The quantity A which appears in this formula is a strictly positive 'redshift factor' which, for time-like trajectories, can be computed from the condition $(vv)=-1$. Taking the square of (4.6), one gets

$$A^{-2}=-[(\eta\eta)+\Omega^2(\xi\xi)]. \tag{4.7}$$

The quantity Ω is the orbital angular velocity of the particle.

Consider a local *stationary observer* whose four-velocity is

$$u^\alpha=\frac{\eta^\alpha}{[(\eta\eta)]^{1/2}}. \tag{4.8}$$

This stationary observer uniquely defines the global rest frame and also the three-dimensional space in terms of the projection tensor

$$h^\alpha_\beta=\delta^\alpha_\beta+u^\alpha u_\beta, \tag{4.9a}$$

and the three-dimensional metric tensor

$$g^*_{ik}=g_{ik}+u_i u_k. \tag{4.9b}$$

(The Latin indices run through 1, 2, 3.) Every geometrical object in the space-time may be projected into the three-dimensional space; for example, the square of the orbital speed $\tilde{v}$, in the three-dimensional space, is equal to the square of the projected velocity given by

$$\tilde{v}^\beta\equiv v^\alpha h^\beta_\alpha=-A\Omega\xi^\beta. \tag{4.10}$$

Taking the square of the velocity $\tilde{v}^\beta$, we arrive at

$$\tilde{v}^2=v^2\gamma^2=\frac{\Omega^2\tilde{r}^2}{1-\Omega^2\tilde{r}^2}, \tag{4.11}$$

where the quantity $\tilde{r}$ is given by

$$\tilde{r}^2\equiv-\frac{(\xi\xi)}{(\eta\eta)}. \tag{4.12}$$

In formula (4.11), $v=\Omega\tilde{r}$ is the 'physical' velocity of the particle and $\gamma=1/\sqrt{1-v^2}$ is the Lorentz γ factor.

The equation $\tilde{r}=$ constant defines the *von Zeipel cylinders* on which the ratio of specific angular momentum to angular velocity (l/Ω) is constant. This follows from the general relation between the angular momentum and angular velocity

$$l=-\frac{(\xi\xi)}{(\eta\eta)}\Omega. \tag{4.13}$$

It is convenient to define a unit-vector $\tilde{\mathbf{q}}$ normal to these cylinders:

$$\tilde{q}_\alpha=\frac{\nabla_\alpha\tilde{r}}{[(\nabla_\beta\tilde{r})(\nabla^\beta\tilde{r})]^{1/2}}. \tag{4.14a}$$

The concept of von Zeipel cylinders was first introduced by Abramowicz (1974) and subsequently proved to be very useful in studying the properties of rotating fluid configurations (see Seguin 1975; Abramowicz & Muchotrzeb 1976; Abramowicz, Jaroszyński & Sikora 1978; Abramowicz 1982; Chakrabarti 1985). In Newtonian theory the families of von Zeipel cylinders and straight cylinder are (of course) identical and therefore one can write the centrifugal force as

$$\mathscr{C}^*=m\frac{v^2}{\mathscr{R}}(\mathbf{q}\boldsymbol{\lambda}^*)=m\frac{v^2}{\mathscr{R}}(\tilde{\mathbf{q}}\boldsymbol{\lambda}^*). \tag{4.14b}$$

In general relativity, however, the two families are distinct and I will show that the direction of the centrifugal force coincides with the outward normal to the von Zeipel cylinders $\tilde{\mathbf{q}}$ and not with the outward normal to the straight cylinders, $\mathbf{q}$.

Let us now compute the acceleration

$$a_\beta\equiv v^\alpha\nabla_\alpha v_\beta. \tag{4.15}$$

We assume that the motion itself is stationary and axially symmetric which means that

$$\eta^\alpha\nabla_\alpha\Omega=0=\xi^\alpha\nabla_\alpha\Omega. \tag{4.16}$$

Using this formula and the formulae for acceleration (4.15) and four-velocity of circular motion (4.6), one obtains

$$a_\beta=\frac{1}{2}\frac{\nabla_\beta(\eta\eta)+\Omega^2\nabla_\beta(\xi\xi)}{(\eta\eta)+\Omega^2(\xi\xi)}. \tag{4.17}$$

The circular free-photon orbit is characterized by $v^\alpha v_\alpha=0$ and $a_\alpha=0$ and, making use of formula (4.17), these two conditions can be written as

$$\Omega=\frac{1}{\tilde{r}},\ \nabla_\alpha\tilde{r}=0. \tag{4.18a}$$

The projection of the photon trajectory into the three-dimensional space is a circle $r=$ constant characterized by

$$\nabla_\alpha\tilde{r}=0. \tag{4.18b}$$

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Particles can move along this trajectory with the use of a rocket engine – only photons can move along it freely. (To avoid a possible misunderstanding let me stress that particles moving along the closed photon trajectory may have physical velocities in the whole range $0 \le v < 1$. There is nothing singular about them, in particular they have finite energies.)

The acceleration formula (4.17) can easily be expressed in another form,

$$a_\beta = \frac{1}{2}\nabla_\beta \ln|(\eta\eta)| - \frac{\tilde{v}^2}{\tilde{r}}\nabla_\beta \tilde{r}. \tag{4.19}$$

Note that the part of the acceleration which depends on the orbital speed is parallel to the normal to the von Zeipel cylinders. This is an important point, which I will make more explicit by introducing a *positive* $\tilde{\mathscr{R}}$ by a formula identical with the Newtonian formula (3.4)

$$\tilde{\mathscr{R}} \equiv \frac{\tilde{r}}{[(\nabla_\beta \tilde{r})(\nabla^\beta \tilde{r})]^{1/2}}. \tag{4.20}$$

$\tilde{\mathscr{R}}$ is the geodesic curvature radius of the $\tilde{r}=$constant circles in the optical reference geometry. An important property of this purely geometrical quantity is that it is infinite at the circular photon path (*cf.* equation 4.18b)

$$\begin{pmatrix}\text{circular photon}\\ \text{path}\end{pmatrix} \Rightarrow (\tilde{\mathscr{R}} = \infty) \tag{4.21a}$$

and the converse implication is also true

$$\begin{pmatrix}\text{circular photon}\\ \text{path}\end{pmatrix} \Leftarrow (\tilde{\mathscr{R}} = \infty). \tag{4.21b}$$

The acceleration formula can now be written as

$$a_\beta = \frac{1}{2}\nabla_\beta \ln|(\eta\eta)| - \frac{\tilde{v}^2}{\tilde{\mathscr{R}}} q_\beta. \tag{4.22}$$

We next project this equation into an arbitrary direction λ^*_β orthogonal to the particle trajectory. The thrust force is $T_\beta = m a_\beta$ and its projection will be denoted by

$$\mathscr{T}^* = m\lambda^*_\beta a^\beta. \tag{4.23}$$

The two terms on the right-hand side of equation (4.22) clearly look like 'gravity' and 'centripetal acceleration' respectively, but I will not *assume* that this is what they are. Instead, I will use the experimental approach, introduced in the previous section,

$$\begin{pmatrix}\text{centrifugal}\\ \text{force}\end{pmatrix} = -\begin{pmatrix}\text{orbital}\\ \text{speed}\end{pmatrix}^2 \frac{\partial(\text{thrust force})}{\partial(\text{orbital speed})^2}, \tag{4.24}$$

in order to determine their physical nature.

From this experimental definition, it follows that the centrifugal force is given by

$$\mathscr{C}^* = m\frac{\tilde{v}^2}{\tilde{\mathscr{R}}}(\lambda^* \tilde{q}). \tag{4.25}$$

Formula (4.25) is *identical* to its Newtonian counterpart (4.14a) and, in addition, all of the quantities are defined in exactly the same way in the Newtonian and relativistic cases. Although this formal identity alone would be a sufficient basis for claiming that formula (4.25) correctly defines the relativistic concept of centrifugal force, I will further support the claim by considering physical consequences of this definition.

First of all, let me point out that from (4.25) it follows that the centrifugal force vanishes independently of v and λ^*_α if and only if $\tilde{\mathscr{R}} = \infty$. We have already said that $\tilde{\mathscr{R}} = \infty$ on the circular path of a free photon (*cf.* equation 4.21b) and so the experimentally defined centrifugal force vanishes for particles moving along this path. It follows that, for such particles, the thrust is independent of the orbital speed (this was first noticed and explained by AL) and so particles which move with different orbital speeds along the circular photon orbit all have the same acceleration which is given by

$$a_\alpha = \frac{1}{2}\frac{\nabla_\alpha(\xi\xi)}{(\xi\xi)}. \tag{4.26}$$

We now prove that a gyroscope spin S_α which is initially parallel to the circular photon path will remain parallel to it when transported along the orbit by an observer with a (general) four-velocity v_α consistent with formula (4.6). If true, this implies that the spin of the gyroscope is always parallel to the Killing vector ξ^α, since the vector ξ^α is itself parallel to circular orbits. The spin of a gyroscope is Fermi–Walker transported along the trajectory (see Synge 1960) and so, because $S_{|\alpha}\xi_{\beta|} = 0$, the result will be proved if we can show that the Killing vector ξ^α is Fermi–Walker transported along the circular free-photon path. This is equivalent to checking whether

$$v^\alpha \nabla_\alpha \xi_\beta = -(a_\beta v^\alpha - v_\beta a^\alpha)\,\xi_\alpha \tag{4.27}$$

on the circular photon orbit. Note, first of all, that from (4.17) it follows that $a^\alpha \xi_\alpha = 0$. The left-hand side of formula (4.27) is

$$A(\eta^\alpha + \Omega\xi^\alpha)\nabla_\alpha \xi_\beta = \tfrac{1}{2}A\Omega\nabla_\beta(\xi\xi) \tag{4.28}$$

and the right-hand side is

$$a_\beta(v\xi) = \frac{1}{2}(r\xi)\frac{\nabla_\beta(\xi\xi)}{(\xi\xi)} = \frac{1}{2}A\Omega\nabla_\beta(\xi\xi). \tag{4.29}$$

Thus, (left-hand side) = (right-hand side) and the result is proved.

We have shown¶¶ that the experimentally defined relativistic centrifugal force shares with its familiar Newtonian prototype the following important dynamical feature: a curve along which particles can move without experiencing a centrifugal force

(i) coincides with a light trajectory in space,

(ii) has the property that gyroscopes transported along it do not precess.

Let me repeat that this is the most important fact which really gives the very essence of the centrifugal force (stressed

¶¶The proof presented in this paper is valid for circular orbits only. Its generalization to arbitrary trajectories is tedious but straightforward and will be given elsewhere.

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by the Huygen's definition) – the force appears (in the instantaneously corotating frame) when the motion of a particle deviates from a "straight' line as established dynamically by light tracing and gyroscopic guiding.

Therefore, dynamically, the situation is exactly the same as in Newtonian theory. However, there is also an important difference since the dynamically defined 'straight' trajectories are, here, not straight in the geometrical sense: they are *not* geodesics in the directly projected three-dimensional space with the metric g^*_{ik}. They became geodesics in the conformally adjusted metric of the three-dimensional space which is, in fact, the ACL *optical reference geometry*:

$$\tilde{g}_{ik}=(\eta\eta)^{-1}g^*_{ik}. \tag{4.30}$$

This can easily be seen from the Fermat principle which states that *in static space-times, light moves between two fixed points in space on a trajectory for which the travel time t is a minimum*. The relativistic Fermat principle was introduced, for static space-times, by Weyl (1917) and is explained in most relativity textbooks – e.g. Landau & Lifshitz (1975) or Adler, Bazin & Schiffer (1965). Its generalization to stationary, non-static space-times was found by Quan (1962). The space-time path of a light ray is given by

$$ds^2=(\eta\eta)(dt^2+d\tilde{l}^2)=0, \tag{4.31}$$

where $d\tilde{l}^2=\tilde{g}_{ik}dx^i dx^k$ is the line element in the optical reference geometry. The Fermat principle requires that

$$\delta\int dt=0,$$

where the integral is taken along the trajectory between the two fixed points. From (4.31) it follows that

$$\delta\int d\tilde{l}=\delta\int(\tilde{g}_{ik}dx^i dx^k)^{1/2}=0. \tag{4.32}$$

The above formula shows that light trajectories in space are geodesic lines in the optical reference geometry. Therefore, in the optical reference geometry the lines of zero centrifugal force have exactly the same properties as in Newtonian theory: they are 'straight' in both the dynamical and geometrical sense. The optical reference geometry is, therefore, naturally connected to the problem and it could be useful for practical purposes.‖‖

Using the optical reference geometry, AP explained why the centrifugal force in circular motion could reverse its sign and become attractive towards the centre of the circle. Their arguments were based on the fact that the direction of the centrifugal force agrees with that of the outward normal to the circle. This vector is orthogonal to the straight-line (photon trajectory) tangent to the circle, and is on the same side of the circle as the tangent line. For the Schwarzschild geometry there exists a closed, circular photon path at $r=r_{\rm ph}$. The light trajectories inside this are more curved than $r=$constant circles and consequently the outward normal to any circle $r=\text{constant}<r_{\rm ph}$ is directed *inwards* towards the centre.

Abramowicz, Miller & Stuchlík (in preparation) show that in static, axially symmetric space-times, the von Zeipel radius $\tilde{r}$ is related to the effective potential for the motion of photons by

$$\tilde{r}^2=\frac{1}{\mathscr{V}_{\rm eff}}. \tag{4.33}$$

The gradient of the effective potential is therefore parallel to the vector $\tilde{\mathbf{q}}$ and so the direction of the centrifugal force (for circular motion) coincides with the direction of the gradient of the effective potential $\mathscr{V}_{\rm eff}$. This has an interesting consequence: the centrifugal force *locally* attracts towards stable circular photon orbits (minima of the effective potential) and repels away from unstable circular photon orbits (maxima of the effective potential). The fact that the centrifugal force repels away from the axis of rotation and attracts towards infinity can also be explained in terms of the gradient of the effective potential.

4.2 Physical approach

Consider two test particles (both with mass m) on the circular photon orbit $\nabla_\alpha\tilde{r}=0$ and two observers comoving with these particles: observer A is stationary, staying at a fixed point of the orbit (his orbital speed is therefore $v_A=0$), and observer B moves around the orbit with orbital speed v_B which is constant in time. To fix the convention, I will assume that velocities are measured in the global rest frame but this is not important for my arguments. Note that neither of the two observers can be in free (geodesic) motion because on this orbit only photons are free. Therefore, they use rocket engines (giving thrusts $\mathscr{T}_A$ and $\mathscr{T}_B$) to move with the nonzero total accelerations a_A and a_B. We are assuming that both of the particles have the same mass m and so

$$\frac{\mathscr{T}_A}{a_A}=\frac{\mathscr{T}_B}{a_B}. \tag{4.34}$$

Suppose that the non-moving (stationary) observer A points his telescope in the direction in which he sees the moving observer B, i.e. the direction tangent to the circular photon path. He may keep this direction fixed by reference to the spin of a gyroscope attached to his telescope. As observer A is *stationary*, the gyroscope will always be tangential to the circular photon path. This means, however, that observer A will actually *see* observer B always at a fixed direction in the sky. He may also use a flash light: if he sends out a light signal which is reflected back by B, because both clockwise and anticlockwise orbiting photons move along the same circle, the signals will be received back by his gyroscopically guided telescope coming from the same direction in which they were sent. Furthermore, because the orbital speed of B is constant in time, A will measure a constant radial speed by Doppler tracking. Therefore, he will conclude that the relative transverse velocity of B is zero and his relative radial velocity is constant. This is equivalent to

‖‖Abramowicz (1990) shows that the optical reference geometry can be based on direct physical measurements of length by the radar time-delay technique. Time is measured by static observers who synchronize their clocks in the standard way (Landau & Lifshitz 1975). Note also that the conformal adjustment employed in defining the optical geometry is used quite often in general relativity. For example, the specific energy *measured* by a stationary observer, $E_0\equiv-(uv)$, must be conformally adjusted by the factor $(\eta\eta)^{1/2}$ to give the conserved quantity $\mathscr{E}$.

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saying that B appears to move with respect to A along a straight line with constant speed, which obviously implies that the relative acceleration between A and B, physically *measured* by A, is zero:

$$a_B - a_A = 0. \tag{4.35}$$

From this it follows that $\mathscr{T}_A = \mathscr{T}_B =$ constant, or that the thrust of the rockets has a fixed value $\mathscr{T}_0$ which does not depend on the orbital speed (this was first noticed by Abramowicz & Lasota 1974, 1977):

$$\mathscr{T} = \mathscr{T}_0 = \text{constant}, \left(\frac{\partial \mathscr{T}}{\partial v^2}\right) = 0. \tag{4.36}$$

Whichever actual definitions of the 'gravitational' and 'centrifugal' forces might be adopted, it is reasonable to argue that (in the instantaneously corotating frame) there are only three forces acting for a general circular motion: the gravitational force $\mathscr{G}$, the centrifugal force $\mathscr{C}$, and the thrust force $\mathscr{T}$, with

$$\mathscr{T} + \mathscr{C} + \mathscr{G} = 0. \tag{4.37}$$

A reasonably defined gravitational force must have the same value, $\mathscr{G} = \mathscr{G}_0$ say, for all particles moving a particular *circular* orbit. A reasonably defined centrifugal force must vanish for a stationary observer, which means that for this observer $\mathscr{T} + \mathscr{G}_0 = 0$. However, from the last formula and from the fact that on the circular photon orbit $\mathscr{T} = \mathscr{T}_0 =$ constant, it immediately follows that $\mathscr{C} = 0$ on the circular photon orbit *independently* of the orbital speed. We conclude that *any* physically reasonable definition of centrifugal force must lead to the result that the centrifugal force vanishes for particles moving along the circular photon orbit, independently of orbital speed. This conclusion does not depend on the choice of reference frame or details of possible definitions of the gravitational and centrifugal forces.

We now argue that a gyroscope spin which is initially parallel to the circular photon orbit will remain parallel to it when the gyroscope moves along the orbit. Suppose that this were not true. In that case, the moving observer B could not see (through his gyroscopically guided telescope) the observer A as always being in the same direction. Thus, he would conclude that the relative acceleration between himself and A is non-zero. This, however, cannot be correct since we have just proved that $a_A = a_B$. This contradiction proves the point.

We have proved that the centrifugal force vanishes when a particle moves along a 'straight' line defined dynamically (by light and gyroscopic tracing). Although we proved this for a closed circular photon path, the proof will be exactly the same for *any* light trajectory in space.*** Thus, exactly as in Newtonian theory, the centrifugal force appears when the motion of a particle deviates from such a *dynamically straight* trajectory in space. In Newtonian theory, a dynamically straight line is also straight in the strictly geometrical sense.

***Note that it was crucial in the proof that the photons move along the same trajectory in both directions. This is true only for strictly static space-times (such as the Schwarzschild geometry) and does not hold in space-times which are only stationary but not static (such as the Kerr geometry). In stationary but non-static space-times, the dragging of inertial frames separates the prograde and retrograde light trajectories.

This can be achieved in general relativity by making a *uniform* rescaling of all distances in the three-dimensional space. In the rescaled geometry (the 'optical reference geometry' of ACL) the light trajectories, the lines defined by non-precessing gyroscopes and the geodesic lines are all the same.

Let us now consider the case of a particle which moves along a non-geodesic trajectory in the optical reference geometry. From our discussion above we know that, in this case, there will be a centrifugal force acting on the particle in its instantaneously corotating reference frame. This force can be *measured* by the experiment (2.12). From the formal discussion in the first part of this section it follows that the result of the experiment gives

$$\mathscr{C}^* = -v^2\left(\frac{\partial \mathscr{T}^*}{\partial v^2}\right) = m\frac{v^2}{\tilde{\mathscr{R}}}\tilde{\mathbf{q}}\cdot\boldsymbol{\lambda}^*. \tag{4.38}$$

This is exactly the same result as in the case of Newtonian theory and all of the symbols used have the same meaning in both theories. I conclude, therefore, that the relativistic concept of centrifugal force is *identical* to the concept familiar from Newtonian theory.

4.3 Reversal of the action of the centrifugal force: the rotosphere

We next consider the particular example of vacuum Schwarzschild space-time, which in spherical coordinates (t, R, θ, φ) has the metric line element

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2(d\theta^2 + \sin^2\varphi). \tag{4.39}$$

Here M is the mass of the central object (not necessarily a black hole). In these coordinates, the Killing vectors η^α and ξ^α are given by

$$\eta_\alpha = \nabla_\alpha t, (\eta\eta) = g_{tt} = -\left(1 - \frac{2M}{R}\right), \tag{4.40a}$$

$$\xi_\alpha = \nabla_{\alpha\varphi}, (\xi\xi) = g_{\varphi\varphi} = R^2\sin^2\theta. \tag{4.40b}$$

It will be convenient to introduce the Schwarzschild tetrad, i.e. the orthonormal set of three vectors $e^{(i)}_\beta$ which have non-zero components

$$e^{(R)}_R = \left(1 - \frac{2M}{R}\right)^{-1/2}, \tag{4.41a}$$

$$e^{(\theta)}_\theta = R, \tag{4.41b}$$

$$e^{(\varphi)}_\varphi = R\sin\theta. \tag{4.41c}$$

The tetrad components of a *space-like* vector x^i are defined by

$$x^{(i)} = e^{(i)}_k x^k. \tag{4.42}$$

I will now discuss the direction of the centrifugal force with respect to the outward direction away from the axis of rotation. In general, the cosine of the angle ε between two space-like unit vectors x^i and y^i is given by

$$\begin{aligned}\cos\varepsilon &= x^{(R)}y^{(R)} + x^{(\theta)}y^{(\theta)} + x^{(\varphi)}y^{(\varphi)}\\ &= x_{(R)}y_{(R)} + x_{(\theta)}y_{(\theta)} + x_{(\varphi)}y_{(\varphi)}.\end{aligned} \tag{4.43}$$

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We have shown that the outward direction away from the axis of rotation is defined by the unit space-like vector q^i, which is normal to the straight cylinders, while the direction of the centrifugal force is determined by the unit space-like vector $\tilde{q}^i$, which is normal to the von Zeipel cylinders.

The straight cylinders are given by the equation $r^2 \equiv (\xi\xi) = \text{constant}$ which, in the case of the Schwarzschild metric, takes the form

$$r^2 = R^2 \sin^2\theta = \text{constant}. \tag{4.44}$$

The vector which defines the outward direction away from the axis of rotation is orthogonal to these cylinders and has the tetrad components

$$q_{(R)} = \frac{\sin\theta(1-2M/R)^{1/2}}{|(1-2M/R\sin^2\theta)^{1/2}|}, \tag{4.45a}$$

$$q_{(\theta)} = \frac{\cos\theta}{|(1-2M/R\sin^2\theta)^{1/2}|}, \tag{4.45b}$$

$$q_{(\varphi)} = 0. \tag{4.45c}$$

The von Zeipel cylinders are given by the equation $\tilde{r}^2 \equiv -(\xi\xi)/(\eta\eta) = \text{constant}$ which, in the case of the Schwarzschild metric, takes the form

$$\tilde{r}^2 = \frac{R^2\sin^2\theta}{(1-2M/R)}. \tag{4.46}$$

The von Zeipel cylinders are shown in Fig. 1. The unit vector which defines the direction of the centrifugal force is orthogonal to the von Zeipel cylinders and has the tetrad components

$$\tilde{q}_{(R)} = \frac{\cos\theta(1-2M/R)^{1/2}}{|[(1-3M/R)^2\sin^2\theta + (1-2M/R)\cos^2\theta]^{1/2}|}, \tag{4.47a}$$

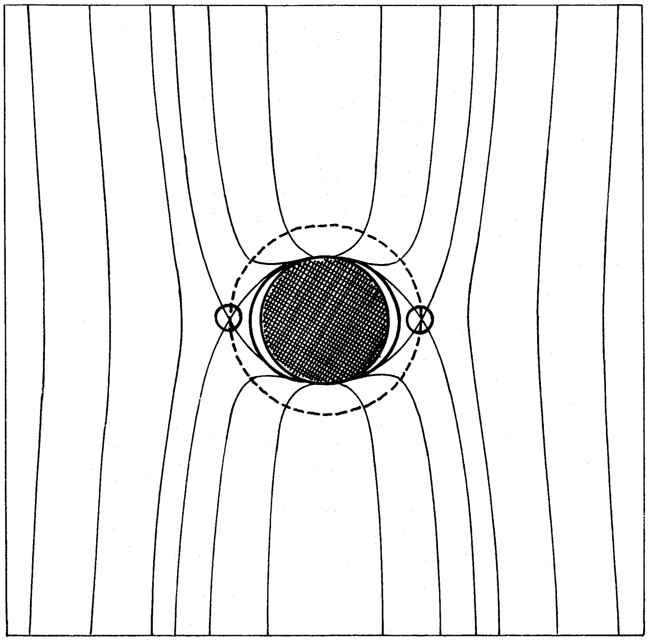

Figure 1. von Zeipel cylinders in the Schwarzschild geometry (solid lines) and the $R=3M$ sphere (dashed line). The shaded circle in the centre represents the Schwarzschild black hole. Two open dots indicate the location of the circular photon at the equatorial plane.

$$\tilde{q}_{(\theta)} = \frac{\sin\theta(1-3M/R)}{|[(1-3M/R)^2\sin^2\theta + (1-2M/R)\cos^2\theta]^{1/2}|}, \tag{4.47b}$$

$$\tilde{q}_{(\varphi)} = 0. \tag{4.47c}$$

The angle ε between the direction of the centrifugal force and the outward direction away from the axis of rotation can be computed from the general formula (4.43) and the formulae (4.45), (4.47) giving

$$\cos\varepsilon(R,\theta) = \left(1 - \frac{3M}{R}\right) Q(R,\theta), \tag{4.48a}$$

where the positive function $Q(R,\theta)$ is given by

$$Q(R,\theta) = \frac{(1-2M/R)^{1/2}}{|1-2M/R\sin^2\theta|^{1/2}} \times [(1-3M/R)^2\sin^2\theta + (1-2M/R)\cos^2\theta]^{-1/2} \tag{4.48b}$$

From this equation it follows that the centrifugal force *attracts* towards the rotation axis, (i.e. $\cos\varepsilon < 0$), in the whole region in which $(R/3M) < \sin^2\theta$. I will call this region *the rotosphere.* Fig. 2 shows the von Zeipel cylinders close to a Schwarzschild black hole and also the rotosphere.

It is obvious that the condition for a rotosphere to exist is that there should be a self-crossing von Zeipel cylinder. Indeed, far away from the gravitating centre, the von Zeipel cylinders almost coincide with the straight cylinders and, in particular, the normal vectors to the two families of cylinders point in almost the same direction. Because the straight cylinders have strictly cylindrical topology, the normal vectors to the two families can point in opposite directions only if the von Zeipel cylinders 'turn inside-out', which is topologically possible only when there is a self-crossing von Zeipel cylinder.††† The self-crossing von Zeipel cylinder for the vacuum Schwarzschild space-time ($\tilde{r} = 3\sqrt{3}M$) can be clearly seen in Fig. 2. At the place where the self-crossing occurs, the normal vector is undefined and so $\nabla_\alpha \tilde{r} = 0$ and $\tilde{\mathscr{R}} = \infty$. This, however, implies (*cf.* 4.21) the existence of a closed photon orbit in space. Indeed, the von Zeipel cylinder $\tilde{r} = 3\sqrt{3}M$ crosses itself along the circle $R = 3M$, $\cos^2\theta = 0$, which corresponds to a closed photon trajectory in the Schwarzschild geometry. Also, it is easy to see that the converse of this implication is true: when there is a closed photon path in space corresponding to an *unstable* circular photon orbit, there must be a self-crossing von Zeipel cylinder and hence a rotosphere must exist.

This can be seen from the following reasoning (Abramowicz, Miller & Stuchlík, in preparation; Abramowicz 1990). Let us consider the motion in space of a photon having energy E and specific angular momentum L. We will denote by V_α the component of the photon's velocity which is orthogonal to both η_α and ξ_α. Let $\tilde{V}^2 = -(\eta\eta)(VV)/\mathscr{L}^2$ (note that $\tilde{V}^2 > 0$). The equation of

†††There is an interesting duality between the regions outside and inside the self-crossing von Zeipel cylinder $\tilde{r} = 3\sqrt{3}$, which was pointed out to me by John Miller: every von Zeipel cylinder with $\tilde{r} > 3\sqrt{3}$ which exists outside the self-crossing cylinder has its 'dual image' inside it. The von Zeipel cylinders for which $\tilde{r} = \infty$ are located at an infinite distance from the axis, outside the $\tilde{r} = 3\sqrt{3}$ surface, and on the black hole horizon inside this surface.

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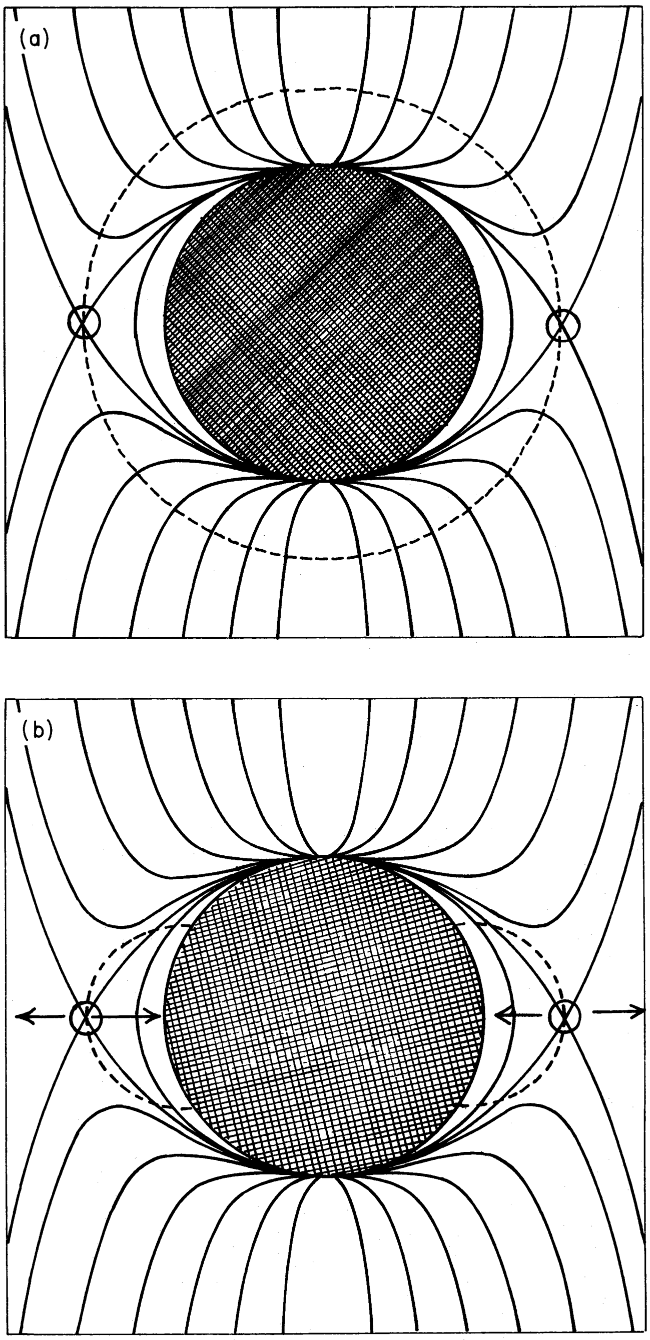

Figure 2. (a) The von Zeipel cylinders (solid lines) and the sphere $R=3M$ (dashed line). The open circle in the centre represents the black hole. This is just a scaled-up version of Fig. 1. (b) The von Zeipel cylinders (solid lines) and the rotosphere (dashed line). The format is the same as that of (a).

motion for the photon can then be written as

$$\tilde{V}^2 = \tilde{b}^2 - \mathscr{V}_{\text{eff}}, \tag{4.49}$$

where $\tilde{b} = \mathscr{E}/\mathscr{L} =$ constant is the impact parameter of the photon ($\tilde{b}=\infty$ for photons crossing the rotation axis). The quantity

$$\mathscr{V}_{\text{eff}} = \frac{1}{\tilde{r}^2} \tag{4.50}$$

is the *effective potential* for the photon motion. Circular photon orbits are located at its extrema, given by $\nabla_\alpha \mathscr{V}_{\text{eff}}=0$, with local maxima corresponding to *unstable* orbits, and local minima corresponding to *stable* ones. Because $\nabla_\alpha \mathscr{V}_{\text{eff}} = -2\tilde{r}^{-3}\nabla_\alpha \tilde{r}$, the vector $\tilde{q}_\alpha$ (the unit vector normal to the von Zeipel cylinders) is not defined at the circular photon orbit:

$$\tilde{q}_\alpha = 0/0. \tag{4.51}$$

This implies that the von Zeipel cylinders have *critical circles* there. Two types of critical circle are possible: the *saddle* type, where a particular von Zeipel cylinder intersects itself, and the *centre* type, in the neighbourhood of which the von Zeipel cylinders have toroidal topology. Because the direction of the vector $\tilde{q}_\alpha$ is the same as the direction of the centrifugal force, the centrifugal force always locally repels away from unstable circular photon orbits and attracts towards stable ones.

This is schematically illustrated in Fig. 3 for the Schwarzschild solution describing both the exterior and the interior of a spherical constant-density star having gravitational radius R_G and radius $R_0<(3/2)R_G$. In Fig. 3(a), the effective potential $\mathscr{V}_{\text{eff}}$ is shown. It has a maximum, corresponding to the familiar unstable circular photon orbit outside the star at $r=r_1\equiv(3/2)R_G$, and a minimum inside the star, corresponding to a stable circular orbit at $r=r_2$. The part corresponding to the interior of the star is shaded. Fig. 3(b) shows an equatorial cross-section of the star with the sections of the von Zeipel cylinders appearing as circles (shown by dashed lines). The two critical circles are shown by heavy lines – the inner circle (centre type) corresponds to the stable circular photon orbit, the outer circle (saddle type) corresponds to the unstable circular photon orbit. The vector $\tilde{q}_\alpha$ is shown by small arrows. Between the two photon orbits the centrifugal force attracts towards the rotation axis. Fig. 3(c) shows a meridional section through the star with the von Zeipel cylinders being represented by heavy lines and the axis of rotation being the straight vertical line in the middle. There are two critical circles: one of the saddle type, which corresponds to the unstable photon orbit (shown by the open dot), and one of the centre type, which corresponds to the stable photon orbit (shown by the solid dot). The equatorial plane is represented by the horizontal dashed line, the interior of the star is shaded and the vector q_α is shown by the small arrows. Note that in the region between the two photon orbits, the force *attracts* towards the axis of rotation.

I will conclude this section with a discussion of the *Ring & String* experiment on the equatorial plane of a Schwarzschild black hole. First, I will write down the expression for the angle $\varepsilon_0 \equiv \varepsilon(R, \theta=\pi/2)$ between the direction of the centrifugal force and the outward direction away from the axis of rotation:

$$\cos\varepsilon_0 = \frac{(1-(3M)/R)}{|(1-(3M)/R)|} = \begin{cases} +1, \text{ if } R>3M \\ -1, \text{ if } R<3M. \end{cases} \tag{4.52}$$

This expression shows clearly that the centrifugal force changes direction at the location of the closed photon orbit: it is repulsive outside this orbit but attractive inside it. The same may be demonstrated by means of the *Ring & String* experiment. The formula for the tension in the strings follows

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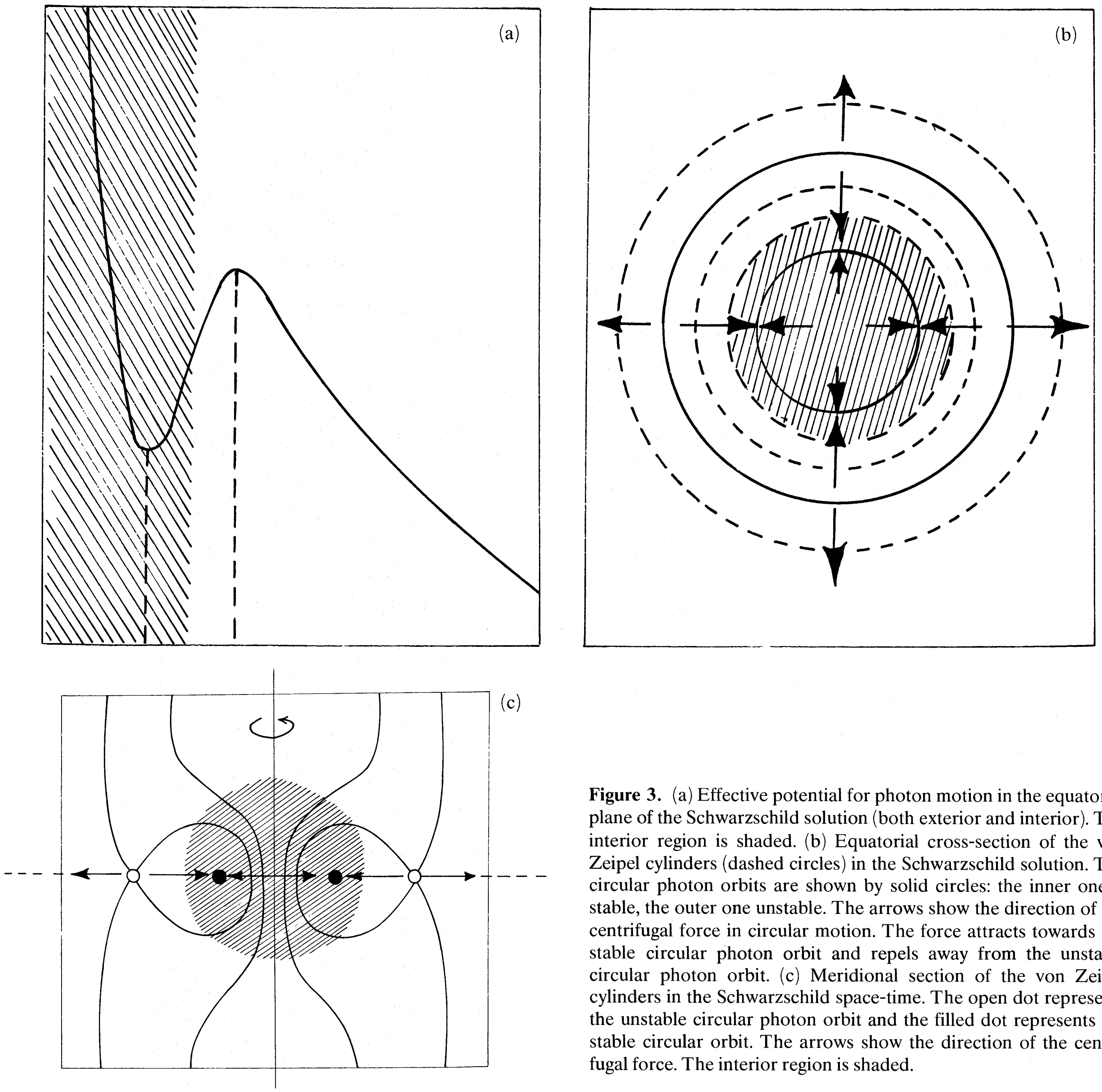

Figure 3. (a) Effective potential for photon motion in the equatorial plane of the Schwarzschild solution (both exterior and interior). The interior region is shaded. (b) Equatorial cross-section of the von Zeipel cylinders (dashed circles) in the Schwarzschild solution. The circular photon orbits are shown by solid circles: the inner one is stable, the outer one unstable. The arrows show the direction of the centrifugal force in circular motion. The force attracts towards the stable circular photon orbit and repels away from the unstable circular photon orbit. (c) Meridional section of the von Zeipel cylinders in the Schwarzschild space-time. The open dot represents the unstable circular photon orbit and the filled dot represents the stable circular orbit. The arrows show the direction of the centrifugal force. The interior region is shaded.

from the acceleration formula (4.22). On the equatorial plane, one has the following expression, in the Schwarzschild tetrad, for the radial component of the tension (which is the only non-zero component):

$$\mathscr{T}_{(R)} = h_0 \left| \frac{M}{R_0} \left(1 - \frac{2m}{R_0}\right)^{-1/2} - \frac{v^2}{R_0} \frac{1}{1-v^2} \left(1 - \frac{3M}{R_0}\right) \right|. \tag{4.53}$$

Here h_0 is a positive constant, which depends on the nature of the strings, and R_0 is the proper circumferential radius of the ring. For $R_0 < 3M$, both terms within the absolute value symbols are *positive*. Therefore, in order to increase the tension, one must *speed up* the rotational velocity of the ring. However, this means, according to our previous discussion of the *Ring & String* experiment, that the centrifugal force is *attractive* towards the axis of rotation. Note that this conclusion does not depend on the actual value of the velocity – in particular it remains true also for velocities close to zero. Formula (4.53) shows clearly that this bizarre result cannot be attributed to some 'repulsive gravitational phenomenon', 'changes of inertia' due to the influence of the Lorentz γ factor, 'interaction of the spin of matter with geometry', or similar concepts.

5 CONCLUSIONS

There are two physically different concepts of centrifugal force (which are, however, equivalent in Newtonian theory). One of these, given by Newton's definition (2.1), attributes the existence of the centrifugal force to the rotation of a reference frame with respect to the global rest frame. The other one, consistent with Huygens' definition (2.4), attri-

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butes the existence of the force (in the instantaneously corotating reference frame) to motion along a curve different from a photon trajectory in space. In a strong gravitational field, closed, exactly circular photon trajectories exist. Therefore, one can introduce a reference frame rotating with respect to the global rest frame, such that a particle with a fixed location in the rotating frame moves along the circular photon trajectory in the global rest frame. In this particular situation, the rotating frame used in Newton's definition is identical to the instantaneously corotating one used in Huygens' definition. According to Newton's definition there should be a centrifugal force acting on the particle, while according to Huygens' definition there should be no centrifugal force in this case. Using formal arguments and discussing several gedanken experiments, I have demonstrated that Huygens' definition is compatible with general relativity, while Newton's definition is not.

When there is an unstable closed photon path in space then, inside the rotosphere, the centrifugal force attracts matter towards the axis of rotation. Within the rotosphere, no circular motion of free particles is possible and matter moving on non-free circular orbits displays very strange and counter-intuitive dynamical behaviour. Earlier attempts to understand such behaviour all missed the point and proposed incorrect or artificial explanations which tended to be very complicated and were often based on only *seemingly* fruitful physical analogies. When I spoke about this subject at a recent seminar, somebody asked me "Could you treat the centrifugal force as a kind of electric force?" My answer was: "Yes, I can. But I prefer to treat the centrifugal force as the centrifugal force!".

ACKNOWLEDGMENTS

I thank many colleagues who have spent long hours discussing with me the issues studied in this paper. First, I am indebted to Dennis Sciama who helped to clarify many of the ideas presented here and pointed out to me that a somewhat similar approach to forces and motion has previously been considered by Synge (1960). Without his help this paper would not have been completed. Jerry Ostriker suggested examining the relation between orbital speed and thrust, which became the main point of the experimental definition of centrifugal force proposed in this paper. George Ellis, Jiři Bičák, Jürgen Ehlers and Donald Lynden-Bell looked critically at the formal details of my arguments. (Jürgen Ehlers, in addition, informed me about several relevant historical references.) S. Chandrasekhar, Roy Kerr, Arieh Königl, Igor Novikov, Bohdan Paczyński, Sterl Phinney, Ivor Robinson, George Rybicki, Bernd Schmidt, Kip Thorne and Andrzej Trautman, suggested some improvements in arguments and presentation. Antonio Lanza has checked most of the relativistic calculations and prepared all the figures. John Miller read all of the various successive versions of the manusript, improving their logical structure and wording. Special thanks are due to Piotr Lasota (our 1974 paper was the beginning of this story) and to other colleagues who have collaborated with me on this subject: Brandon Carter, A. R. Prasanna, John Miller, Božena Muchotrzeb-Czerny and Zdenek Stuchlik.

This research was supported by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica.

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